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ORIGINAL ARTICLE

H_∞ Fuzzy Control of Semi-Markov Jump Nonlinear Systems under σ-Error Mean Square Stability

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This paper is concerned with the problem of time-varying H_∞ fuzzy control for a class of semi-Markov jump nonlinear systems in the sense of σ-error mean square stability. The nonlinear plant is described via the Takagi-Sugeno fuzzy model. By defining a time-varying mode-dependent Lyapunov function, a set of sufficient stability and stabilization criteria for non-disturbance case is first derived and then applied to the investigation of H_∞ performance analysis and H_∞ fuzzy controller design problems of semi-Markov jump nonlinear systems. Different from the traditional stochastic switching system framework, the probability density function of sojourn time is exploited to circumvent the complex computation of transition probabilities. The derived conditions can cover the time-invariant mode-dependent and time-invariant mode-independent H_∞ fuzzy control schemes as special cases. A classic cart-pendulum system is presented to demonstrate the effectiveness and advantages of the proposed theoretical results.

Keywords: H_∞ control; stochastic systems; σ-error mean square stability; semi-Markov jump nonlinear systems; time-varying controller

1. Introduction

Robust control of general nonlinear systems has always been a valuable and yet challenging topic for the lack of a unified mathematical framework (Gao, Wang, & Wang, 2005). Takagi-Sugeno (T-S) fuzzy model has fixed structure and can effectively approximate most nonlinear systems. Some remarkable results can be found in (Chen, Lam, & Lam, 2015; S. Y. Xu & Lam, 2005). However, it should be noted that the stochastic instantaneous change of system dynamics, caused by changes in work conditions such as weather, airspeed (Boukas, 2007), or the equipment (e.g., switches, oscillators (Torikai & Saito, 1998), voltage feedback buck converter, and hyperchaos generator (Takahashi, Nakano, & Saito, 2004)) cannot be accommodated by this kind of model. Therefore, the study of fuzzy stochastic switching system is highly motivated (Li, Chen, Zhou, & Qian, 2009; Sheng, Gao, Zhang, & Chen, 2015).

As one of the most important stochastic switching systems, Markov jump systems (MJSs) have received extensive attention in the past few decades. It has been widely applied in many practical plants, such as power systems (Willsky & Levy, 1979), aerospace systems (Kiyak, Çetin, & Kahvecioğlu, 2008), electronic systems (Ma, Kawakami, & Tse, 2004), and networked control systems (Zhang, Gao, & Kaynak, 2013). Many significant results for MJSs have been reported; the readers are referred to (Costa, do Val, & Geromel, 1999; S. Xu, Chen, & Lam, 2003; Zhang & Lam, 2010) and the references therein. It should be mentioned that the key characteristics and also the main restriction of MJSs is the memoryless property of transition probabilities (TPs), which requires that the distribution of sojourn time (the interval between two consecutive jumps of system mode) of each system mode should be subject to geometric distributions for

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discrete-time domain (or exponential distributions for continuous-time domain). This requirement, however, is not always realistic.

In response to such a challenge, semi-Markov jump systems (S-MJSs), in which the TPs of system modes are time-varying and sojourn-time dependent, have been proposed and discussed. Because the TPs for S-MJSs at each time k have memory, infinite iterations are needed to obtain the full information of TPs (Howard, 1964). Therefore, even for the stability and stabilization problems, investigations on S-MJSs are quite limited and existing results are usually obtained using certain hidden assumptions on the bounds of TPs or the types of sojourn-time distributions. See Hou, Luo, Shi, and Nguang (2006); Huang and Shi (2011, 2013a, 2013b); Lee, Ma, Xu, and Ju (2015); Schwartz (Northwestern University, 2003); Shen, Su, and Park (2016); Shen, Wu, and Park (2015) for example. An approach based on the semi-Markov kernel is used by Zhang, Leng, and Colaneri (2016) to handle the stabilization problem in the sense of σ-error mean square stability (σ-MSS [1]). The authors employed a sojourn-time distribution which depended not only on the current system mode but also on the next mode that the system will jump to. In the present paper, we will extend this work using the time-varying Lyapunov function approach for the purpose of reducing the conservatism. Meanwhile, since the theorems are to be derived based on the probability density function (PDF) of sojourn time, knowledge of the time-dependent TPs at each time k is no longer necessary, which makes the results more reasonable.

This paper focuses on the H_∞ stability and stabilization problems for the discrete-time fuzzy S-MJSs. First, the sufficient stability and stabilization conditions are discussed for the non-disturbance fuzzy S-MJS in the sense of σ-MSS. Then, the time-varying Lyapunov function approach is applied to the H_∞ stability and stabilization problems and a feasible time-varying fuzzy controller is obtained which will guarantee the σ-MSS of the underlying system with a disturbance attenuation index. It is then demonstrated that the proposed results can be easily extended to cover time-invariant mode-dependent and time-invariant mode-independent control schemes and thus are more general. A practical example is presented to illustrate the effectiveness and feasibility of the derived theoretical results.

Notations: In this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the n-dimensional Euclidean space and the set of all $n \times m$ real matrices, respectively. $\mathbb{N}$ and $\mathbb{N}_+$ denote the sets of non-negative integers and positive integers, respectively. The subscript of a set denotes an additional constraint on the set, for example, $\mathbb{N}_{[s_1,s_2]} \triangleq \{k \in \mathbb{R} | s_1 \le k \le s_2\}$. For the notation $(\Psi, \mathcal{F}, \Pr)$, Ψ represents the sample space, $\mathcal{F}$ is the σ-algebra of subsets of the sample space, and $\Pr$ is the probability measure on $\mathcal{F}$. X' and $\mathbb{E}[X]$ stand for the transpose and mathematical expectation of matrix X, respectively. The space of square summable infinite sequence is denoted as $l_2[0,\infty)$. The notation $P > 0 (\ge 0)$ means P is real symmetric positive (semi-positive) definite. In addition, $diag\{\cdots\}$ and $diag_{(n)}\{X\}$ stand for a block-diagonal matrix and an $n \times n$ block-diagonal matrix where all diagonal entries are X, respectively. The symbol $*$ is used as an ellipsis for the terms that are introduced by symmetry. **I** and **0** represent the identity matrix and zero matrix, respectively. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

2. Preliminaries and Problem Formulation

Fix the complete probability space $(\Psi, \mathcal{F}, \Pr)$ and consider the following discrete-time fuzzy S-MJS:

Rule i: IF $\zeta_1(k)$ is μ_{i1} and $\zeta_2(k)$ is μ_{i2} and ... and $\zeta_m(k)$ is μ_{im} THEN

$$x(k+1)=A(i,r(k))x(k) + B(i,r(k))u(k) + E(i,r(k))w(k) \tag{1}$$

$$y(k)=C(i,r(k))x(k) + D(i,r(k))u(k) + F(i,r(k))w(k) \tag{2}$$

where $i \in \mathcal{R} \triangleq \{1, 2, \ldots, M_R\}$, M_R is the number of the IF-THEN rules; $\zeta(k) = [\zeta_1(k), \zeta_2(k), \ldots, \zeta_p(k)]$ is the premise variable; $\mu_{i1}, \mu_{i2}, \ldots, \mu_{ip}$ are the fuzzy sets; $x(k) \in \mathbb{R}^{n_x}$, $y(k) \in \mathbb{R}^{n_y}$, $u(k) \in \mathbb{R}^{n_u}$, and $w(k) \in \mathbb{R}^{n_w}$ with $\{w(k)\}_{k\in\mathbb{N}} \in l_2[0,\infty)$ are the system state, system output, control input and external

[1] In this paper, we will slightly abuse MSS as the abbreviation of either mean square stability or mean square stable.

disturbance input, respectively. $\{r(k)\}_{k\in\mathbb{N}}$ is a stochastic process and considered to be a semi-Markov chain in this paper, which takes values in a finite set $\mathcal{I} \triangleq \{1, 2, \ldots, M_I\}$ and governs the switching among M_I system modes. For simplicity of presentation, the following notation $A_{ir_k} = A(i, r(k))$ has been used throughout the paper. Therefore, for $r(k) = p \in \mathcal{I}$, the set of matrices of the p^{th} system mode for the i^{th} rule is denoted by $(A_{ip}, B_{ip}, E_{ip}, C_{ip}, D_{ip}, F_{ip})$, which are real known matrices. It is assumed that the premise variables do not depend on the input signal $u(k)$. Then, given a pair of $(x(k), u(k))$, a more compact presentation of the discrete-time T-S fuzzy S-MJS is given by

$$\begin{aligned} x(k+1) &= \sum\nolimits_{i=1}^{M_R} h_i(k)\{A_{ir_k}x(k) + B_{ir_k}u(k) + E_{ir_k}w(k)\}, \\ y(k) &= \sum\nolimits_{i=1}^{M_R} h_i(k)\{C_{ir_k}x(k) + D_{ir_k}u(k) + F_{ir_k}w(k)\}, r(k) \in \mathcal{I} \end{aligned} \tag{3}$$

where $h_i(k) \triangleq v_i(\zeta(k))/\sum_{i=1}^{M_R} v_i(\zeta(k))$, $v_i(\zeta(k)) \triangleq \prod_{l=1}^{p} \mu_{il}(\zeta_l(k))$, and $\mu_{il}(\zeta_l(k))$ is the membership degree of $\zeta_l(k)$ in μ_{il}. More generally, we assume that $v_i(\zeta(k)) \geq 0, i \in \mathcal{R}$ and $\sum_{i=1}^{M_R} v_i(\zeta(k)) > 0$ are satisfied, which can result in that $0 \leq h_i(k) \leq 1, i \in \mathcal{R}$.

To introduce the semi-Markov chain (SMC) formally, we shall recall the concept of Markov renewal chain (MRC), for which the following three stochastic processes are first needed (more details can be found in Barbu and Limnios (2006) and the references therein):

(i) The stochastic process $\{R_n\}_{n\in\mathbb{N}_+}$ taking values in $\mathcal{I}$, where R_n is the index of system mode at the n^{th} jump and $R_0 \in \mathcal{I}$ is the initial state.

(ii) The stochastic process $\{k_n\}_{n\in\mathbb{N}_+}$ taking values in $\mathbb{N}_+$, where k_n denotes the time at the n^{th} jump. It is noted that $k_0 = 0$, and k_n increases monotonically with n.

(iii) The stochastic process $\{S_n\}_{n\in\mathbb{N}_+}$ taking values in $\mathbb{N}_+$, where $S_n = k_n - k_{n-1}, \forall n \in \mathbb{N}_+$ denotes the sojourn time of mode R_{n-1} between the $(n-1)^{th}$ jump and n^{th} jump, and $S_0 = 0$.

Definition 1: (Barbu & Limnios, 2006) The stochastic process $\{(R_n, k_n)\}_{n\in\mathbb{N}_+}$ is said to be a discrete-time homogeneous MRC if the following holds $\forall p, q \in \mathcal{I}, \forall \tau \in \mathbb{N}_+$ and $\forall n \in \mathbb{N}_+$:

$$\begin{aligned} \Pr(R_{n+1} = q, S_{n+1} = \tau | R_0, \ldots, R_n = p; k_0, \ldots, k_n) &= \Pr(R_{n+1} = q, S_{n+1} = \tau | R_n = p) \\ &= \Pr(R_1 = q, S_1 = \tau | R_0 = p). \end{aligned}$$

In addition, from Barbu and Limnios (2006), $\{R_n\}_{n\in\mathbb{N}_+}$ is called the embedded Markov chain (EMC) of MRC $\{(R_n, k_n)\}_{n\in\mathbb{N}_+}$, and the transition probability matrix (TPM) $\Theta = [\theta_{pq}]_{p,q\in\mathcal{I}}$ of $\{R_n\}_{n\in\mathbb{N}_+}$ is defined by $\theta_{pq} \triangleq \Pr(R_{n+1} = q | R_n = p), \forall p, q \in \mathcal{I}$ with $\theta_{pp} = 0$.

With the above concepts, the definition of SMC is given as below.

Definition 2: (Barbu & Limnios, 2006) Consider an MRC $\{(R_n, k_n)\}_{n\in\mathbb{N}_+}$. The chain $\{r(k)\}_{k\in\mathbb{N}_+}$ is said to be an SMC associated with MRC $\{(R_n, k_n)\}_{n\in\mathbb{N}_+}$, if $r(k) = R_{N(k)}, \forall k \in \mathbb{N}_+$, where $N(k) \triangleq \max\{n \in \mathbb{N} | k_n \leq k\}$.

Since the stochastic variable varies with the jump instant k_n for EMC, but with the sampling instant k for SMC, the two stochastic chains are much different. For an SMC, one can define the following probabilities.

Definition 3: (Barbu & Limnios, 2006) For a given SMC $\{r(k)\}_{k\in\mathbb{N}_+}$, we have:

(i) The PDF of the sojourn time depending on the current and the next system modes of the SMC is defined as:

$$\omega_{pq}(\tau) \triangleq \Pr(S_{n+1} = \tau | R_{n+1} = q, R_n = p)$$

with $\forall q \neq p, p, q \in \mathcal{I}, \forall \tau \in \mathbb{N}_+$ and $\omega_{pp}(\tau) = 0, \forall p \in \mathcal{I}, \forall \tau \in \mathbb{N}_+$.

(ii) The semi-Markov kernel $\Pi(\tau) = (\pi_{pq})_{p,q\in\mathcal{I}} \in \mathbb{R}^{M_I\times M_I}$ is defined as

$$\pi_{pq}(\tau) \triangleq \Pr(R_{n+1} = q, S_{n+1} = \tau | R_n = p) = \theta_{pq}\omega_{pq}(\tau)$$

with $\forall q \neq p, p, q \in \mathcal{I}, \forall\tau \in \mathbb{N}_+$ and $\pi_{pp}(\tau) = 0, \forall p \in \mathcal{I}, \forall\tau \in \mathbb{N}_+$.

To focus on the stochastic characteristics, we assume that the sojourn time cannot reach infinity. That is to say, $\eta_{pp}(\infty) \triangleq 1 - \sum_{S_n\in\mathbb{N}_+}\sum_{q\neq\in\mathcal{I}}\pi_{pq}(S_n) = 0, p \in \mathcal{I}$ holds. Now, the stability and H_∞ performance definitions are presented for better illustrating the purposes of this paper.

Definition 4: (Zhang et al., 2016) Given an upper bound of the sojourn time $T^p_{\max} \in \mathbb{Z}_{\geq 1}, \forall p \in \mathcal{I}$, system (3) is said to be σ-error Mean Square Stable (σ-MSS) if for $u(k) \equiv 0$ and any initial condition $x_0 \in \mathbb{R}^{n_x}, r_0 \in \mathcal{I}$, the following equation holds:

$$\lim_{k\to\infty} E\left[\|x(k)\|^2\right]\Big|_{x_0,r_0,S_0,\{S_{n+1}\leq T^p_{\max}|R_n=p,p\in\mathcal{I},n\in\mathbb{N}_+\}} = 0 \tag{4}$$

with $\sigma \triangleq \sum_{p=1}^{M_I}|\ln(F_p(T^p_{\max}))|$ where $F_p(\tau) = \Pr(S_{n+1} \leq \tau | R_n = p) = \sum_{l=0}^{\tau}\sum_{q\in\mathcal{I}}\pi_{pq}(l)$. $F_p(\tau), p \in \mathcal{I}, \tau \in \mathbb{N}_+$ is called the cumulative density function (CDF) of the sojourn time for system mode p and it is assumed that $F_p(0) = \sum_{q\in\mathcal{I}}\omega_{pq}(0) = 0$.

Definition 5: For a constant $\gamma > 0$, system (3) is said to be σ-MSS and has an H_∞ disturbance attenuation performance index γ if the system is σ-MSS and under zero initial condition $E\{\|y(k)\|_2\} < \gamma\|w(k)\|_2$ where $\|w(k)\|_2 \triangleq \sqrt{\sum_0^\infty w'(k)w(k)}$ holds for all nonzero $w(k) \in l_2[0,\infty)$.

For the nonlinear system (3), the parallel distributed compensation (PDC) scheme will be used to design the state feedback fuzzy controller. Due to the existence of the stochastic processes, both the time-varying and mode-dependent ideas will be applied in the design of the PDC fuzzy controller of the following form:

$$u(k) = \sum_{i=1}^{M_R} h_i(k)K_i(r(k),\delta(k))x(k), r(k) \in \mathcal{I} \tag{5}$$

with $\delta(k) \triangleq k - k_{N(k)}$. Therefore, for $r(k) \in \mathcal{I}$, the closed-loop system can be obtained:

$$\begin{aligned} x(k+1) &= \sum_{i=1}^{M_R}\sum_{j=1}^{M_R} h_i(k)h_j(k)\left\{\left(A_{ir_k} + B_{ir_k}K_j(r(k),\delta(k))\right)x(k) + E_{ir_k}w(k)\right\}, \\ y(k) &= \sum_{i=1}^{M_R}\sum_{j=1}^{M_R} h_i(k)h_j(k)\left\{(C_{ir_k} + D_{ir_k}K_j(r(k),\delta(k)))x(k) + F_{ir_k}w(k)\right\}. \end{aligned} \tag{6}$$

Then, the problem to be addressed in this paper is to design a mode-dependent time-varying fuzzy controller in the form of (5), such that the resulting closed-loop fuzzy S-MJS (6) is σ-MSS with H_∞ disturbance attenuation performance.

3. Main Results

In this section, we first propose the sufficient stability and stabilization criteria for the underlying system with $w(k) \equiv 0$. Based on these results, the fuzzy controller can be designed such that the closed-loop system is σ-MSS with an H_∞ performance. To begin with, the following lemma which will be used in the proof of our main results is presented.

Lemma 3.1: *(Guan & Chen, 2004) For any real matrices $X_i, i = 1, 2, \ldots, M_R$ and $P > 0$ with appropriate dimensions, one has*

$$\sum_{i=1}^{M_R}\sum_{j=1}^{M_R} h_i h_j X_i' P X_j \leq \sum_{i=1}^{M_R} h_i X_i' P X_i$$

where $h_i \geq 0$ for $i = 1, 2, \ldots, M_R$ and $\sum_{i=1}^{M_R} h_i = 1$.

3.1. *σ-error mean square stability analysis and stabilization*

In the next theorem, a sufficient σ-MSS criterion will be presented which is testable for a class of fuzzy S-MJS with $u(k) \equiv 0$ and $w(k) \equiv 0$.

Theorem 3.2: *Consider system (3) with $u(k) \equiv 0$ and $w(k) \equiv 0$. Then, system (3) is σ-MSS, if for all $p \in \mathcal{I}$, there exist $T_{max}^p \in \mathbb{N}_+$ and a set of symmetric matrices $\{P_i(p, \vartheta) > 0\}$, $i \in \mathcal{R}, \vartheta \in \mathbb{N}_{[0,T_{max}^p]}$ such that for all $i, j \in \mathcal{R}, p, q \in \mathcal{I}, \vartheta \in \mathbb{N}_{[1,T_{max}^p]}$, (7) and (8) hold.*

$$A'(i,p)P_j(p,\vartheta)A(i,p) - P_i(p,\vartheta-1)<0 \tag{7}$$

$$\sum_{\vartheta=1}^{T_{max}^p} \sum_{q\neq p, q\in\mathcal{I}} \eta_{pq}(\vartheta)\left\{P_i(q,0) - P_i(p,\vartheta)\right\}<0 \tag{8}$$

where $\eta_{pq}(\vartheta) \triangleq \pi_{pq}(\vartheta)/\rho_p(\vartheta)$ with $\rho_p(\vartheta) \triangleq \sum_{\vartheta=1}^{T_{max}^p} \sum_{q\neq p,q\in\mathcal{I}} \pi_{pq}(\vartheta)$.

Proof. Consider the time-varying stochastic Lyapunov function

$$V\left(x(k), r(k), \delta(k)\right)|_{r_k=p}=x'(k)\sum_{i=1}^{M_R} h_i(k)P_i\left(r(k)=p, \delta(k)\right)x(k), k \in \mathbb{N}_{(k_{n-1},k_n]}.$$

If $x(k) = 0$ for some finite k, the σ-MSS will be guaranteed. Therefore, it is reasonable to assume that $x(k) \neq 0$. Then, along the solution of the unforced system (3), one has the following relationship for $r(k) = p \in \mathcal{I}, k \in \mathbb{N}_{[k_n,k_{n+1}-2]}$:

$$\begin{aligned}
&V\left(x(k+1), r(k+1), \delta(k+1)\right) - V\left(x(k), r(k), \delta(k)\right)\\
=&x'(k+1)\sum_{j=1}^{M_R} h_j(k+1)P_j(p,\delta(k+1))x(k+1) - x'(k)\sum_{i=1}^{M_R} h_i(k)P_i\left(p,\delta(k)\right)x(k)\\
=&\sum_{i=1}^{M_R}\sum_{g=1}^{M_R}\sum_{j=1}^{M_R} h_i(k)h_g(k)h_j(k+1)x'(k)\left[A_{ip}'P_j(p,\delta(k+1))A_{gp} - P_i(p,\delta(k))\right]x(k)\\
\leq&\sum_{i=1}^{M_R}\sum_{j=1}^{M_R} h_i(k)h_j(k+1)x'(k)\left[A_{ip}'P_j(p,\delta(k+1))A_{ip} - P_i(p,\delta(k))\right]x(k).
\end{aligned}$$

Then, it follows from (7) that $\forall r(k) \in \mathcal{I}, k \in \mathbb{N}_{[k_n,k_{n+1}-2]}$, it is guaranteed that

$$V\left(x(k+1), r(k+1), \delta(k+1)\right) - V\left(x(k), r(k), \delta(k)\right)<0. \tag{9}$$

Since (7) holds for $p \in \mathcal{I}, \vartheta \in \mathbb{N}_{[1,T_{max}^p]}$, one can derive that $\forall n \in \mathbb{N}_+$

$$V\left(x(k_{n+1}), r(k_{n+1}-1), k_{n+1}-k_n\right) - V\left(x(k_{n+1}-1), r(k_{n+1}-1), \delta(k_{n+1}-1)\right)<0. \tag{10}$$

Then, combining (9) and (10), it can be inferred that

$$\begin{aligned}
&V\left(x(k_{n+1}), r(k_{n+1}-1), k_{n+1}-k_n\right) - V\left(x(k_n), r(k_n), \delta(k_n)\right)\\
=&V\left(x(k_{n+1}), r(k_n), S_{n+1}\right) - V\left(x(k_n), r(k_n), 0\right) < 0.
\end{aligned} \tag{11}$$

On the other hand, it follows from (8) that

$$
\begin{aligned}
&\mathbb{E}\left[V\left(x(k_{n+1}), r(k_{n+1}), \delta(k_{n+1})\right)\right]\big|_{x_{k_n}, r(k_n)} - V\left(x(k_n), r(k_n), \delta(k_n)\right) \\
=&\mathbb{E}\left[V\left(x(k_{n+1}), r(k_{n+1}), 0\right) - V\left(x(k_n), r(k_n), 0\right)\right]\big|_{x_{k_n}, r_{k_n}} \\
\leq&\mathbb{E}\left[V\left(x(k_{n+1}), r(k_{n+1}), 0\right) - V\left(x(k_{n+1}), r(k_n), S_{n+1}\right)\right]\big|_{x_{k_n}, r_{k_n}} \\
=&x'(k_{n+1})\sum\nolimits_{i=1}^{M_R} h_i(k_{n+1})\sum\nolimits_{S_{n+1}=1}^{T_{max}^p}\sum\nolimits_{q\neq p, q\in\mathcal{I}}\eta_{pq}(S_{n+1}) \\
&\times\left\{P_i(q,0) - P_i(p, S_{n+1})\right\}x(k_{n+1}) < 0.
\end{aligned} \tag{12}
$$

Without loss of generality, it can be assumed that there exists a set of positive matrices $Q(i), i \in \mathcal{I}$ such that

$$
\mathbb{E}\left[V\left(x(k_{n+1}), r(k_{n+1}), \delta(k_{n+1})\right)\right]\big|_{x_{k_n}, r_{k_n}} - V\left(x(k_n), r(k_n), \delta(k_n)\right) \leq -x'(k_n)Q(i)x(k_n).
$$

Since $\mathbb{E}\left[V\left(x(k_{n+1}), r(k_{n+1}), \delta(k_{n+1})\right)\right]\big|_{x_{k_n}, r_{k_n}} > 0$, it is straightforward to show that $V(x(k_n), r(k_n), \delta(k_n)) \geq x'(k_n)Q(i)x(k_n)$. Therefore, it can be proved that there exists a constant $\varepsilon \in (0,1)$ such that $\forall x(k_n) \neq 0$

$$
\begin{aligned}
&\frac{\mathbb{E}\left[V\left(x(k_{n+1}), r(k_{n+1}), \delta(k_{n+1})\right)\right]\big|_{x_{k_n}, r_{k_n}}}{V(x(k_n), r(k_n), \delta(k_n))} - \frac{V\left(x(k_n), r(k_n), \delta(k_n)\right)}{V(x(k_n), r(k_n), \delta(k_n))} \\
\leq&-\frac{x'(k_n)Q(i)x(k_n)}{V\left(x(k_n), r(k_n), \delta(k_n)\right)} \leq \varepsilon - 1.
\end{aligned} \tag{13}
$$

It can be inferred from (12) that

$$
\mathbb{E}\left[V\left(x(k_{n+1}), r(k_{n+1}), \delta(k_{n+1})\right)\right]\big|_{x_{k_n}, r_{k_n}} \leq \varepsilon V\left(x(k_n), r(k_n), \delta(k_n)\right). \tag{14}
$$

Then, one has

$$
\mathbb{E}\left[V\left(x(k_n), r(k_n), \delta(k_n)\right)\right]\big|_{x_{k_0}, r_{k_0}} \leq \varepsilon^n V\left(x(k_0), r(k_0), \delta(k_0)\right).
$$

Summing from $n = 0$ to $n = N$, one can obtain that

$$
\begin{aligned}
\mathbb{E}\left[\sum\nolimits_{n=0}^{N} V\left(x(k_n), r(k_n), \delta(k_n)\right)\right]\Bigg|_{x_{k_0}, r_{k_0}} \leq&\left(1 + \varepsilon + \varepsilon^2 + \ldots + \varepsilon^N\right)V\left(x(k_0), r(k_0), \delta(k_0)\right) \\
=&\frac{\left(1 - \varepsilon^N\right)}{1 - \varepsilon}V\left(x(k_0), r(k_0), \delta(k_0)\right).
\end{aligned}
$$

When N tends to ∞, we can further show that

$$
\lim_{N\to\infty}\mathbb{E}\left[\sum\nolimits_{n=0}^{N} V\left(x(k_n), r(k_n), \delta(k_n)\right)\right]\Bigg|_{x_{k_0}, r_{k_0}} \leq \frac{1}{1 - \varepsilon}V\left(x(k_0), r(k_0), \delta(k_0)\right)
$$

which implies that

$$
\begin{aligned}
\lim_{N\to\infty}\mathbb{E}\left[\sum\nolimits_{n=0}^{N} x'(k_n)x(k_n)\right]\Bigg|_{x_{k_0}, r_{k_0}} \leq&\frac{x'(k_0)\sup_{m\in\mathcal{R}} P_m\left(r(k_0), \delta(k_0)\right)x(k_0)}{\inf_{m\in\mathcal{R}, n\in\mathbb{Z}, r_{k_n}\in\mathcal{I}}\left\{\lambda_{\min}\left(P_m(r(k_n), \delta(k_n))\right)\right\}(1 - \varepsilon)} \\
\triangleq&\widetilde{P}\left(\varepsilon, r(k_0), x(k_0), \delta(k_0)\right).
\end{aligned}
$$

Because $0<\varepsilon<1$, we have $\widetilde{P}\left(\varepsilon, r(k_0), x(k_0), \delta(k_0)\right)>0$ as a bounded positive matrix. Thus, based on the definition of limitation, it can be concluded that $\lim_{n\to\infty}\mathbb{E}\left[x'(k_n)x(k_n)\right]=0$. Due to $\lim_{n\to\infty}k_n=\infty$, the previous equation $\lim_{n\to\infty}\mathbb{E}\left[x'(k_n)x(k_n)\right]=0$ is equivalent to $\lim_{k\to\infty}\mathbb{E}\left[x'(k)x(k)\right]=0$, which implies (4). □

Based on Theorem 3.2, the existence condition of stabilizing state-feedback fuzzy controller (5) for fuzzy S-MJS (3) with $w(k)\equiv 0$ is presented in the following theorem.

Theorem 3.3: *Consider system (3) with $w(k)\equiv 0$. The underlying system is σ-MSS if there exist a set of symmetric matrices $H_i(p,\vartheta)>0$, a set of matrices $U_i(p,\varrho)$, and a matrix Z, $i\in\mathcal{R}, p\in\mathcal{I}, \vartheta\in\mathbb{N}_{[0,T^p_{max}]}, \varrho\in\mathbb{N}_{[1,T^p_{max}]}, \vartheta,\varrho\in\mathbb{N}_+$, such that the inequalities (15)-(16) hold for all $p,q\in\mathcal{I}, i,j,g\in\mathcal{R}$, and $\vartheta\in\mathbb{N}_{[1,T^p_{max}]}$.*

$$\begin{bmatrix} H_j(p,\vartheta)-Z-Z' & A(i,p)Z+B(i,p)U_g(p,\vartheta-1) \\ * & -H_i(p,\vartheta-1)\end{bmatrix}<0, \tag{15}$$

$$\sum\nolimits_{\vartheta=1}^{T^p_{max}}\sum\nolimits_{q\neq p,q\in\mathcal{I}}\eta_{pq}(\vartheta)\{H_i(q,0)-H_i(p,\vartheta)\}<0 \tag{16}$$

where $\eta_{pq}(\vartheta)$ is defined in (8). Moreover, the admissible controller gain for (5) is given by $K_g(p,\vartheta)=U_g(p,\vartheta)Z^{-1}$.

Proof. Perform congruence transformations on (15) by $diag\{Z^{-1},Z^{-1}\}$ and set $P_i(p,\vartheta)=(Z')^{-1}H_i(p,\vartheta)Z^{-1}$. Since $(Z')^{-1}H_j(p,\vartheta)Z^{-1}-Z^{-1}-(Z')^{-1}=P_j(p,\vartheta)-Z^{-1}-(Z')^{-1}>-(Z')^{-1}P_j^{-1}(p,\vartheta)Z^{-1}$ is guaranteed by $(P_j(p,\vartheta)-Z^{-1})'P_j^{-1}(p,\vartheta)(P_j(p,\vartheta)-Z^{-1})$, it can be inferred from (15) that

$$\begin{bmatrix} -(Z')^{-1}P_j^{-1}(p,\vartheta)Z^{-1} & (Z')^{-1}\mathbb{K}(i,p,g,\vartheta-1) \\ * & -P_i(p,\vartheta-1)\end{bmatrix}<0 \tag{17}$$

with $\mathbb{K}(i,p,g,\vartheta-1)\triangleq A(i,p)+B(i,p)K_g(p,\vartheta-1)$. Then, according to the Schur complement lemma, one can prove that (17) is equivalent to (7) by performing congruence transformations with $diag\{P_j(p,\vartheta)Z,I\}$. This completes the proof. □

3.2. H_∞ performance analysis and controller design

To proceed further, the corresponding criterion dealing with the H_∞ control problem for the fuzzy S-MJS can be derived based on the previous results.

Theorem 3.4: *Consider system (3) with $u\equiv 0$. The closed-loop system (6) is σ-MSS with an H_∞ performance γ, if for all $p\in\mathcal{I}$, there exist $T^p_{max}\in\mathbb{N}_+$ and a set of symmetric matrices $\{P_i(p,\vartheta)>0\}$, $i\in\mathcal{R}, \vartheta\in\mathbb{N}_{[0,T^i_{max}]}$ such that for all $i,j\in\mathcal{R}, p,q\in\mathcal{I}, \vartheta\in\mathbb{N}_{[1,T^i_{max}]}$, (18) and (19) hold.*

$$\begin{bmatrix} -I & 0 & C(i,p) & F(i,p) \\ * & -P_j(p,\vartheta) & P_j(p,\vartheta)A(i,p) & P_j(p,\vartheta)E(i,p) \\ * & * & -P_i(p,\vartheta-1) & 0 \\ * & * & * & -\gamma^2 I\end{bmatrix}<0 \tag{18}$$

$$\sum\nolimits_{\vartheta=1}^{T^p_{max}}\sum\nolimits_{q\neq p,q\in\mathcal{I}}\eta_{pq}(\vartheta)\{P_i(q,0)-P_i(p,\vartheta)\}<0 \tag{19}$$

Proof. Based on the property of negative matrix, it can be inferred from (18) that

$$A'(i,p)P_j(p,\vartheta)A(i,p)-P_i(p,\vartheta-1)+C'(i,p)C(i,p)<0. \tag{20}$$

Thus, according to Theorem 3.2, the σ-MSS of the closed-loop system (6) is guaranteed by (20) and (19). By Schur complement lemma, one can see that (18) is equivalent to the following inequality.

$$\begin{bmatrix} -P_i(p,\vartheta-1) & 0 \\ * & -\gamma^2 I \end{bmatrix} + \begin{bmatrix} C(i,p) & F(i,p) \\ P_j(p,\vartheta)A(i,p) & P_j(p,\vartheta)E(i,p) \end{bmatrix}' \times \begin{bmatrix} I & 0 \\ * & P_j^{-1}(p,\vartheta) \end{bmatrix} \begin{bmatrix} C(i,p) & F(i,p) \\ P_j(p,\vartheta)A(i,p) & P_j(p,\vartheta)E(i,p) \end{bmatrix} < 0 \tag{21}$$

By pre- and post-multiplying $[x'(k_n+\vartheta-1); w'(k_n+\vartheta-1)]$ and $[x'(k_n+\vartheta-1); w'(k_n+\vartheta-1)]'$ to (21), it can be derived that

$$\begin{aligned} & x'(k_n+\vartheta-1)(C'(i,p)C(i,p)+A'(i,p)P_j(p,\vartheta)A(i,p)-P_i(p,\vartheta-1))x(k_n+\vartheta-1) \\ & +2x'(k_n+\vartheta-1)(C'(i,p)F(i,p)+A'(i,p)P_j(p,\vartheta)E(i,p))w(k_n+\vartheta-1) \\ & +w'(k_n+\vartheta-1)(F'(i,p)F(i,p)+E'(i,p)P_j(p,\vartheta)E(i,p)-\gamma^2 I)w(k_n+\vartheta-1) \\ = & y'(k_n+\vartheta-1)y(k_n+\vartheta-1)-\gamma^2 w'(k_n+\vartheta-1)w(k_n+\vartheta-1) \\ & +x'(k_n+\vartheta)P_j(p,\vartheta)x(k_n+\vartheta)-x'(k_n+\vartheta-1)P_i(p,\vartheta-1)x(k_n+\vartheta-1)<0. \end{aligned}$$

Assume that $r_{k_n}=p, r_{k_{n+1}}=q$. Then, summing from $\vartheta=1$ to S_{n+1}, one can deduce

$$\begin{aligned} & \sum_{\vartheta=1}^{S_{n+1}} \big\{y'(k_n+\vartheta-1)y(k_n+\vartheta-1)-\gamma^2 w'(k_n+\vartheta-1)w(k_n+\vartheta-1) \\ & +x'(k_n+\vartheta)P_j(p,\vartheta)x(k_n+\vartheta)-x'(k_n+\vartheta-1)P_i(p,\vartheta-1)x(k_n+\vartheta-1)\big\} \\ = & \sum_{k=k_n}^{k_{n+1}-1} \big\{y'(k)y(k)-\gamma^2 w'(k)w(k)\big\}+x'(k_{n+1})P_j(p,S_{n+1})x(k_{n+1}) \\ & -x'(k_n)P_i(p,0)x(k_n)<0. \end{aligned}$$

Due to $\sum_{\vartheta=1}^{T^p_{max}}\sum_{q\neq p,q\in\mathcal{I}}\eta_{pq}(\vartheta)=1$ and $\eta_{pq}(\vartheta)\geq 0$, one has

$$\begin{aligned} & \sum_{S_{n+1}=1}^{T^p_{max}}\sum_{q\neq p,q\in\mathcal{I}}\eta_{pq}(S_{n+1})\Big\{\sum_{k=k_n}^{k_{n+1}-1}\big[y'(k)y(k)-\gamma^2 w'(k)w(k)\big] \\ & +x'(k_{n+1})P_j(p,S_{n+1})x(k_{n+1})-x'(k_n)P_i(p,0)x(k_n)\big\} \\ = & \sum_{k=k_n}^{k_{n+1}-1}\big[y'(k)y(k)-\gamma^2 w'(k)w(k)\big]+\sum_{S_{n+1}=1}^{T^p_{max}}\sum_{q\neq p,q\in\mathcal{I}}\eta_{pq}(S_{n+1})x'(k_{n+1}) \\ & \times P_j(p,S_{n+1})x(k_{n+1})-x'(k_n)P_i(p,0)x(k_n)<0. \end{aligned}$$

According to (19), the following inequality can be guaranteed by the above inequality:

$$\begin{aligned} \sum_{k=k_n}^{k_{n+1}-1}\big[y'(k)y(k)-\gamma^2 w'(k)w(k)\big]+\sum_{S_{n+1}=1}^{T^p_{max}}\sum_{q\neq p,q\in\mathcal{I}}\eta_{pq}(S_{n+1})x'(k_{n+1}) \\ \times P_j(q,0)x(k_{n+1})-x'(k_n)P_i(p,0)x(k_n)<0. \end{aligned}$$

Additionally, since $\sum_{j\in\mathcal{R}}h_j(\vartheta)=1, h_j(\vartheta)\geq 0$, we have

$$\begin{aligned} \sum_{k=k_n}^{k_{n+1}-1}\big[y'(k)y(k)-\gamma^2 w'(k)w(k)\big]+\sum_{S_{n+1}=1}^{T^p_{max}}\sum_{q\neq p,q\in\mathcal{I}}\eta_{pq}(S_{n+1})x'(k_{n+1}) \\ \times\sum_{j\in\mathcal{R}}h_j(k_{n+1})P_j(q,0)x(k_{n+1})-x'(k_n)\sum_{i\in\mathcal{R}}h_i(k_n)P_i(p,0)x(k_n)<0. \end{aligned}$$

Therefore, if we set $\mathcal{V}_{k_n}(x(k_n), r_{k_n}) \triangleq x'(k_n)\sum_{i\in\mathcal{R}}\{h_i(k_n)P_i(r_{k_n}, 0)\}x(k_n)$, it can be derived that

$$J \triangleq \mathbb{E}\left\{\|y\|_2^2 - \gamma^2\|w\|_2^2\right\} = \mathbb{E}\left\{\sum_{n=0}^{\infty}\sum_{k=k_n}^{k_{n+1}-1}(y'(k)y(k) - \gamma^2 w'(k)w(k))\right\}$$
$$\leq \sum_{n=0}^{\infty}\mathbb{E}\left\{\sum_{k=k_n}^{k_{n+1}-1}\left(y'(k)y(k) - \gamma^2 w'(k)w(k) + \mathcal{V}_{k_{n+1}}(x(k_{n+1}), r_{k_{n+1}}) - \mathcal{V}_{k_n}(x(k_n), r_{k_n})\right)\right\} \leq 0.$$

Thus, the proof is completed. □

Theorem 3.5: *Consider system (3) with the controller in the form of (5). The closed-loop system (6) is σ-MSS with an H_∞ performance γ, if for all $p \in \mathcal{I}$, there exist $T^p_{max} \in \mathbb{N}_+$ and a set of symmetric matrices $\{H_i(p,\vartheta) > 0\}$, $U_i(p,\vartheta)$ and Z where $i \in \mathcal{R}, \vartheta \in \mathbb{N}_{[0,T^i_{max}]}$, such that for all $i, j, g \in \mathcal{R}, p, q \in \mathcal{I}, \vartheta \in \mathbb{N}_{[1,T^i_{max}]}$, (22) and (23) hold.*

$$\begin{bmatrix} -I & 0 & C(i,p)Z + D(i,p)U_g(p,\vartheta-1) & F(i,p) \\ * & \mathbb{H}(j,p,\vartheta) & A(i,p)Z + B(i,p)U_g(p,\vartheta-1) & E(i,p) \\ * & * & -H_i(p,\vartheta-1) & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0 \tag{22}$$

$$\sum\nolimits_{\vartheta=1}^{T^p_{max}}\sum\nolimits_{q\neq p, q\in\mathcal{I}}\eta_{pq}(\vartheta)\left\{H_i(q,0) - H_i(p,\vartheta)\right\} < 0 \tag{23}$$

with $\mathbb{H}(j,p,\vartheta) \triangleq H_j(p,\vartheta) - Z' - Z$. Further, the admissible controller gain is given by $K_i(p,\vartheta) = U_i(p,\vartheta)Z^{-1}$.

Proof. The proof is similar to that of Theorem 3.3 and is thus omitted here. □

Remark 1: Different from the previous work of Zhang et al. (2016), the stability analysis is based on the sample time k instead of jump instant k_n. The latter cannot be used directly to discuss the H_∞ performance. On the other hand, the time-varying H_∞ control is for the first time studied herein for the underlying systems. The time-varying H_∞ control can cover the time-invariant H_∞ control as a special case and thus is more general.

Remark 2: The time-invariant mode-dependent and time-invariant mode-independent controllers can be easily derived based on Theorem 3.5 via replacing $U_i(p,\vartheta)$, $K_i(p,\vartheta)$ with $U_i(p)$, $K_i(p)$ and U_i, K_i, respectively, which indicates that these two control schemes can be well deemed as special cases of the proposed time-varying mode-dependent scheme. The comparison among different control schemes will be given in the following numerical example.

4. Numerical Example

In this example, the H_∞ stabilization problem for the cart-pendulum system illustrated in Fig. 1 will be considered to demonstrate the validity and applicability of the developed theoretical results. As shown in Gao and Chen (2007), the motion of the pendulum can be described as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{g\sin(x_1) - \alpha m l x_2^2 \sin(2x_1)/2 - \alpha\cos(x_1)u}{4l/3 - \alpha m l\cos^2(x_1)} \end{aligned}$$

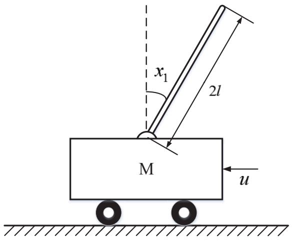

Figure 1.: Inverted pendulum on a cart

where $x_1 \in [-\pi/2, \pi/2]$ denotes the angle of the pendulum; x_2 denotes the angular velocity of the pendulum; u is the force applied to the cart; m is the mass of the pendulum; M is the mass of the cart; $\alpha = 1/(m+M)$ is a known constant; and $2l$ is the length of the pendulum. The gravity constant is $g = 9.8m/s^2$. Here the system parameters are set as $m = 2.0kg, M = 8.0kg, 2l = 1.0m$.

We assume that the data observed contain two kinds of uncertainties: one is the external disturbance $w(k)$ belonging to the l_2 domain, and the other is a kind of stochastic uncertainties governed by a semi-Markov process. Here, the second kind is introduced by the angular velocity measurement, and the errors between the observed and real angular velocities can be approximately partitioned into two classes: 5% and 10% smaller than the real data. To model this uncertainty, the error percentage at each sampling time can be described by a semi-Markov stochastic variable with two states where mode 1 means 5% and mode 2 indicates 10%. That is to say, we have $x_2(k) = 1.05\bar{x}_2(k)$ for mode 1 and $x_2(k) = 1.1\bar{x}_2(k)$ for mode 2, where $\bar{x}_2(k)$ is the measured angular velocity. Then, by the first-order Euler approximation approach (Yang, Zhang, Wang, & Gao, 2013), the whole system can be approximated by the following two-rule T-S fuzzy S-MJS:

Plant Rule 1: **IF** $x_1(k)$ is about 0, **THEN**

$$\begin{aligned} x(k+1) &= A_1(r(k))x(k) + B_1(r(k))u(k) + E_1(r(k))w(k), \\ y(k) &= C_1(r(k))x(k) + D_1(r(k))u(k) + F_1(r(k))w(k) \end{aligned}$$

Plant Rule 2: **IF** $x_1(k)$ is about $\pm\pi/2$, **THEN**

$$\begin{aligned} x(k+1) &= A_2(r(k))x(k) + B_2(r(k))u(k) + E_2(r(k))w(k), \\ y(k) &= C_2(r(k))x(k) + D_2(r(k))u(k) + F_2(r(k))w(k) \end{aligned}$$

where

$$\begin{aligned} A_1(1) &= \begin{bmatrix} 1 & (1+0.05)T \\ \frac{gT}{4l/3-\alpha ml} & 1 \end{bmatrix}, A_2(1) = \begin{bmatrix} 1 & (1+0.05)T \\ \frac{2gT}{\pi(4l/3-\alpha ml\beta^2)} & 1 \end{bmatrix}, \\ A_1(2) &= \begin{bmatrix} 1 & (1+0.1)T \\ \frac{gT}{4l/3-\alpha ml} & 1 \end{bmatrix}, A_2(2) = \begin{bmatrix} 1 & (1+0.1)T \\ \frac{2gT}{\pi(4l/3-\alpha ml\beta^2)} & 1 \end{bmatrix}, \\ B_1(1) &= B_1(2) = \begin{bmatrix} 0 \\ -\frac{\alpha T}{4l/3-\alpha ml} \end{bmatrix}, B_2(1) = B_2(2) = \begin{bmatrix} 0 \\ -\frac{\alpha\beta T}{4l/3-\alpha ml\beta^2} \end{bmatrix}, \\ C_1(1) &= C_2(1) = C_1(2) = C_2(2) = [T\ 0], D_1(1) = D_2(1) = D_1(2) = D_2(2) = 0, \\ E_1(1) &= E_2(1) = E_1(2) = E_2(2) = [T\ \ T]', F_1(1) = F_2(1) = F_1(2) = F_2(2) = T \end{aligned}$$

with $T = 0.01s$ being the sample time, $\beta = \cos(88^\circ)$. Since the errors between the measured angular velocity and the real angular velocity are described as the uncertainties of system matrices, $\bar{x}_2(k)$ can be

Table 1.: Indexes γ^* and σ for different cases

(T^1_{max}, T^2_{max})	γ^* for Case 1	γ^* for Case 2	γ^* for Case 3	σ
(2,2)	0.5034	0.5041	0.5056	1.2909
(2,5)	0.5037	0.5046	0.5060	1.1809
(2,10)	0.5037	0.5047	0.5060	1.1803
(2,20)	0.5037	0.5047	0.5060	1.1803
(3,3)	0.5029	0.5040	0.5052	0.4486
(3,5)	0.5030	0.5041	0.5053	0.4271
(3,10)	0.5030	0.5041	0.5053	0.4265
(3,20)	0.5030	0.5041	0.5053	0.4265
(5,5)	0.5026	0.5038	0.5049	0.0109
(5,10)	0.5026	0.5038	0.5049	0.0103
(5,20)	0.5026	0.5038	0.5049	0.0103

written as $x_2(k)$ without losing the generality. The membership functions are given as

$$\mu_1(k) = \begin{cases} \frac{2}{\pi}x_1(k) + 1 & x_1(k) \in (-\frac{\pi}{2}, 0] \\ -\frac{2}{\pi}x_1(k) + 1 & x_1(k) \in (0, -\frac{\pi}{2}) \end{cases}$$
$$\mu_2(k) = 1 - \mu_1(k)$$

The sojourn time follows the Bernoulli distribution $\omega_{12}(\tau) = 0.6^\tau \cdot 0.4^{(5-\tau)} \cdot 5!/((5-\tau)!\tau!)$ and the Weibull distribution $\omega_{21}(\tau) = 0.4^{(\tau-1)^{1.3}} - 0.4^{\tau^{1.3}}$.

First, we will compare the time-varying mode-dependent (case 1), the time-invariant mode-dependent (case 2), and the time-invariant mode-independent (case 3) H_∞ control schemes. By solving the conditions in Theorem 4, one can obtain the minimized feasible γ^* for case 1. Then, the corresponding minimized feasible γ^* for cases 2 and 3 can be calculated by replacing $U_i(p,\vartheta), K_i(p,\vartheta)$ in Theorem 4 with $U_i(p), K_i(p)$ and U_i, K_i, respectively. As shown in Table 1, one can note that the index γ^* for case 1 is less than those for the other two cases under the same situation, which illustrates that the time-varying mode-dependent H_∞ control scheme is less conservative than the time-invariant mode-dependent and the time-invariant mode-independent H_∞ control schemes in the sense of the disturbance attenuation performance level. Meanwhile, between any two sets of (T^1_{max}, T^2_{max}), the difference in γ^* is monotonously increasing with the increase of the difference in σ. Further, γ^* decreases with the growth in T^1_{max} or reduction in T^2_{max}, which suggests that the system described by mode 1 can be more easily stabilized than that described by mode 2. This observation can be explained by the fact that the eigenvalues for $A_1(1), A_2(1)$ are closer to 1 than those for $A_1(2), A_2(2)$.

Second, to further demonstrate the effectiveness of the derived results, we assume the external disturbance input $w(k) = 0.5\exp(-5k)\sin(k)$ and $T^1_{max} = 5, T^2_{max} = 10$. Then, one can simulate the state response $x(k)$ and output signal $y(k)$ under zero initial condition, as shown in Fig. 2. The control input $u(k)$ is depicted in Fig. 3 and the control scheme is illustrated in Fig. 4. The circular symbol describes the system mode at time k, and the triangle and the square depict the active controller $K(p,\delta(k)) = \{K_1(p,\delta(k)), K_2(p,\delta(k))\}, p = 1,2, k \in \mathbb{N}$. If a set of controllers is not active, $\delta(k)$ will be "Not Active". Since $\delta(k)$ depends on the sampling instant k, the control scheme is time-varying. It can be computed that the index γ is about 0.2041, less than the derived minimum feasible index $\gamma^* = 0.5026$, which further verifies the validity and applicability of the developed theoretical results.

5. Conclusion

In this paper, new results are developed for the H_∞ control of a class of semi-Markov jump nonlinear systems in the sense of σ-MSS. The T-S fuzzy model is employed to model the nonlinear plant. By constructing a time-varying mode-dependent Lyapunov function, a time-varying mode-dependent fuzzy control scheme is proposed, which reduces the conservatism of the derived criteria compared with the time-invariant Lya-

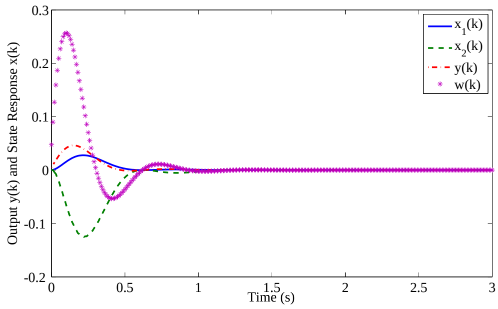

Figure 2.: Output and state response of the inverted pendulum system with $T^1_{max} = 5, T^2_{max} = 10$

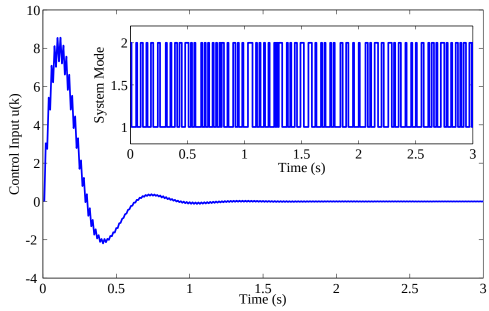

Figure 3.: Control input of the inverted pendulum system with $T^1_{max} = 5, T^2_{max} = 10$

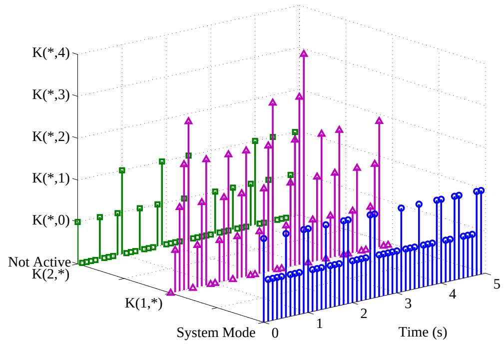

Figure 4.: Illustration of the time-varying control scheme

punov function approach. Based on the derived theorems, the parallel results for the time-invariant mode-dependent and time-invariant mode-independent H_∞ fuzzy control are also obtained, which evidences the advantages of the developed results by theoretical analysis. The effectiveness of the proposed theory is further verified via numerical simulation of a classic cart-pendulum system.

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